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THE DEFINITION OF INFINITY

WE are again reminded by Professor Cobb[1] of the importance of coming to some conclusion with respect to the validity of the modern conception of the mathematical infinite. As my contribution to the discussion I offer some considerations in support of the contention that the "New Infinite" as defined by Dedekind and Cantor is a doubly ambiguous conception. This double ambiguity is disclosed by an examination (1) of the notion of "similarity" or "one-to-one correspondence," and (2) of that of "totality," as these notions are employed in the definition. Let us consider each of these points in turn.

1. *Infinity and one-to-one correspondence.*—The definition of an infinite aggregate or system given by Cantor and Dedekind depends upon the notion of "equal power," "equivalence," or "similarity." "Aggregates with finite cardinal numbers," says Cantor,[2] "are called 'finite aggregates'; and all others are called 'transfinite aggregates,' and their cardinal numbers 'transfinite cardinal numbers.'" The infinite or "transfinite" numbers and aggregates are thus defined negatively, as those which are *not finite;* and we must, accordingly, seek the distinguishing mark of the finite number. This is contained in the theorem that "If M is an aggregate such that it is of equal power with none of its parts, then the aggregate (M, e), which arises from M by the addition of a single new element e, has the same property of being of equal power with none of its parts." This theorem is used in establishing the fundamental properties of the "unlimited series of finite cardinal numbers," and thus becomes a virtual part of their definition. Finite aggregates, accordingly, are never equivalent to any of their parts, while infinite aggregates may be. "The first example of a transfinite aggregate," continues Cantor, "is given by the totality of finite cardinal numbers."

Dedekind's definition, although verbally different, is in substance the same. It runs as follows: "A system S is said to be *infinite* when it is similar to a proper part of itself; in the contrary case, S is said to be a finite system."[3] The point is, of course, not merely that two systems which are assumed or already known to be infinite are similar or one-to-one correspondent, *even if* one is only a part of the other. That such a similarity or equivalence of whole and part is to be found was the very puzzle that had perplexed the older mathematicians. The achievement of Dedekind (if it be a genuine achievement) is rather the reversal of the method of attack. The

[1] This JOURNAL, Vol. XIV., p. 688.

[2] *The Theory of Transfinite Number,* Jourdain's translation, p. 103.

[3] *Essays on the Theory of Numbers,* Beman's translation, p. 41.

"similarity" of a whole to its "proper part" is no longer merely an observed fact, nor is it for him an inference from their infinity; but infinity is now *defined to be* such similarity.

In Dedekind's terminology every system is a part of itself; while a system which contains some, but not all, of the elements of a given system is a *proper* part of the given system. Any two systems or aggregates are said to stand to each other in the relation of "one-to-one correspondence" when for each element or term of one there is one and only one term of the other. Mr. Russell's illustration is familiar: "The relation of father to son is called a one-many relation, because a man can have only one father, but may have many sons; conversely, the relation of son to father is called many-one. But the relation of husband to wife (in Christian countries) is called one-one, because a man can not have more than one wife, or a woman more than one husband."[4]

Now it is easy to show that these definitions contain a very grave ambiguity. For *whenever a series is found that is similar to, that is to say, in one-to-one correspondence with, a proper part of itself,* the series in question *may be shown to be in several other* kinds of correspondence *with the same part;* in fact, any sort of correspondence that one pleases to look for may be discovered; and, furthermore, any scheme or plan of correspondence may be shown to be just as rigidly *determined by law* as any other—specifically, as the scheme of one-to-one correspondence, which some partisans of the "new infinite" have too hastily assumed to be *the* relation in which the two series eternally stand.

Consider as a typical case the favorite, not to say hackneyed, example of a part which is similar to its whole, namely, the series of even numbers in relation to the series which contains all the numbers, odd as well as even. By definition, the series of even numbers is a proper part of the series of whole numbers, and yet by the law that each of its terms is a number twice the corresponding term of the series of whole numbers, it is required to stand in one-to-one correspondence with that series; and, therefore, by Dedekind's definition, the series of whole numbers is an infinite system. But we find that any other correspondence than the one-to-one may be seen, if we wish to see it. This may be exhibited thus:

I.	(W)	1,	2,	3,	4,	5,	6,	...
	(P)	2,	4,	6,	8,	10,	12,	...

II.	(W)	1,	2,	3,	4,	5,	...
	(P)	2,4,	6,8,	10,12,	14,16,	18,20,	...

[4] *Scientific Method in Philosophy,* p. 203.

III.	(W)	1,2,	3,4,	5,6,	7,8,	...
	(P)	2,	4,	6,	8,	...

IV.	(W)	1,2,	3,4,	5,6,	7,8,	...
	(P)	2,4,6,	8,10,12,	14,16,18,	20,22,24,	...

Case I. is the case which has been supposed to be *the* situation. In the other three cases we have respectively a one-to-two, a two-to-one, and a two-to-three correspondence. Now these other sorts of correspondence are determined by clear and definite rules of exactly the same kind as, although a little more complicated than, the rule which determines the one-to-one correspondence. In Case II. let the rule be that the second of the two terms paired with any one term of the whole series shall be four times that term; in Case III. the second of the two terms of (W) is the same number as the one term of (P) with which the two terms of (W) are bound up; in IV. every two terms of (W) are bound up with three of (P), and the rule determining the correspondence is that the last term of any given group of (P) shall be *three* times the last term of the corresponding group of (W). Now it is necessary to insist that the (P) of I., of II., of III., and of IV. is exactly the same series; for the "proper part" 2, 4, 6, 8, 10, *etc.*, is the part that is considered in each case. It has been shown, then, that the whole series stands to this proper part in these *various* relations of correspondence *in exactly the same sense* in which it stands to it in the relation of *one-to-one* correspondence.

The demonstration that this is true of any "proper part" of the series of whole numbers that one chooses to consider, as, for example, the series of multiples by 3, 4, *etc.*, or the series of the squares, of the cubes, *etc.*, of the terms of the natural series of numbers, must be left to the ingenuity and patience of the reader. He will find that the correspondence of a whole and a proper part of itself, which has been taken as the essential notion in the new definition of infinity, turns out, when more closely scrutinized, to be a nose of wax; it can be bent in any direction that one pleases.

What then do we mean when we say that two series are "similar" to each other? Do we mean (1) that the whole "system" and its "proper part" stand to each other in a relation of one-to-one correspondence *and in no other*, or (2) that they are in one-to-one correspondence and *also* related to each other in accordance with *other* schemes of correspondence?

This ambiguity in the meaning of "similarity" has given rise to some clever juggling with the conception of *equality*. Thus Professor Keyser assures us that it is a great error to suppose that the whole-part axiom is universally valid; that it ought rather to be considered as a "logical blade" which divides the finite from the in-

finite. He even discovers an analogy to the doctrine of the Trinity in the relation of the even numbers E, the odd numbers O, and the rational fractions F, to the manifold of all the rational numbers M; for "we have here *three* infinite manifolds E, O, F, no two of which have so much as a single element in common, and yet the three together constitute one manifold M exactly equal in wealth of elements to each of its infinite components."[5] The analogy manifestly depends upon a definition of equality: to be "equal in wealth of elements" is the same thing as to be "similar." So far as I know, Mr. Bertrand Russell has not shown any interest in Trinitarian apologetics. But like Professor Keyser he identifies "equality" with "similarity" or one-to-one correspondence. He tells us that without referring to the census we know that the number of English wives is exactly equal to the number of English husbands.[6] Professor Royce illustrates the same point by referring his readers to a company of marching soldiers, each of whom is seen to carry *one* gun. Even without counting, he says, we know that the number of soldiers is equal to the number of guns.[7]

The difficulty is that neither husbands, wives, soldiers, nor guns are infinitely numerous; and, while one-to-one correspondence may be accepted as a criterion of equality or even regarded as the meaning of equality in the case of *finite* collections, when we seek in the manner suggested by these examples to assure ourselves of the numerical equality of infinite series, the argument breaks down. If "equality" is no more than a relation of one-to-one correspondence, then of course by Dedekind's definition of an infinite system, such a system must be *equal* to a part of itself. The entire series of rational numbers is then "equal in wealth of elements" to the series of odd numbers or to the series of even numbers, and each of these to the other. But, in view of the fact, pointed out above, that when infinite series are found to be in one-to-one correspondence, they may also be shown to be in any other sort of correspondence that one chooses to look for, there is no more reason for regarding such series as numerically equal than there is for saying that one is twice or three or any number of times as rich in elements as another; if a one-to-one correspondence proves that two given series are equal, then a *two*-to-one correspondence ought to prove that one is *twice* the other, *etc.*; and if we were able at will to shift our point of view so as to see two soldiers carrying one gun or one soldier carrying two guns, or if the discovery of monogamy, polygamy, or polyandry in England depended merely upon the caprice of the observer, then we should

[5] *The New Infinite and the Old Theology*, pp. 85 ff.

[6] *Scientific Method in Philosophy*, p. 203.

[7] *Hibbert Journal*, I., pp. 37 ff.

certainly know nothing whatever about the relative abundance of guns and soldiers or of husbands and wives.

The source of the confusion is clear. If "similarity" is to be regarded as logically equivalent to "equality," then it must be interpreted in our first sense; that is to say, similar collections must be understood to be such as stand to one another in the relation of one-to-one correspondence *and no other.* In the nature of the case, however, no two infinite collections can be show to be similar in this sense. Accordingly, when the method of comparison which is now in question is carried over from finite to infinite collections, similarity must needs be understood in the second sense, as meaning one-to-one correspondence along with *other* relations of correspondence; but it is evident that in this sense similarity is not the same as equality.

It is, then, highly desirable that the champions of the "new infinite" should tell us clearly in what sense they understand the notion of "similarity." "A system S is *infinite* if it is similar to a proper part of itself." Does this mean that the whole and the proper part are in an *exclusively one-to-one* correspondence, or that the one-to-one correspondence is only one of the many relations of correspondence which subsist between the given collections? If the former is the correct interpretation of the definition, then, so far as I am aware, no genuine example of an infinite system has ever been adduced. At any rate, no example of an infinite system is revealed by an examination of the mutual relations of the various series of cardinal numbers. Accordingly, if this is the meaning of "similarity," the class of all classes each of which is similar to a proper part of itself is a *class without any members.* On the other hand, if the latter is understood to be the meaning of the definition, if the whole and its proper part are in a relation of one-to-one correspondence, and *also* in relations of one-to-two correspondence, two-to-three correspondence, *etc.,* then, to be sure, there *are* infinite systems. But then we are not justified in regarding the subsistence of a one-to-one correspondence between two infinite series as a proof of their equality; and, unless the fact that the part in question *is a part*—that is to say, is included within, but not coextensive with the system to which it belongs—be taken as evidence that it is *less* than its whole, it is meaningless to speak of any quantitative comparison whatsoever between the whole and a proper part of itself.

2. *The New Infinite and the Notion of Totality.*—In whichever sense the notion of "similarity" be taken, the "new" definition of infinity is logically implied by the old definition of the infinite as the *endless;* for any endless series is *inexhaustible,* and between two inexhaustible series it is always possible to exhibit a one-to-one correspondence, or any sort of correspondence that one chooses to look for,

inasmuch as, however far the pairing of terms or the correlation of groups may be carried, there can never be any dearth of partners or of groups of terms in either series. Accordingly the so-called new infinite may not unfairly be said to be no more than the old infinite in disguise. Consequently it can not be supposed to escape the logical difficulties which beset the older notion. In particular the new formulation does not remove the self-contradiction from the idea of a "realized infinite." This conception remains open to the fatal objection urged by Renouvier[8] that "the completed synthesis (*sommation effectuée*) of a series which by hypothesis is endless (*interminable*) is a contradiction in terms."

The new definition has, indeed, the appearance of avoiding the self-contradiction in the conception of the realized infinite. Thus Dedekind's "discovery" is hailed by Keyser as "one of the greatest achievements in the history of thought."[9] In the opinion of Russell, the new notion of infinity clears up all the puzzles in the conception of the continuum, and makes it unnecessary to seek for a finitist theory of the world.[10] And the new definition was eagerly grasped by Royce to save his Absolute Self from the criticism of Bradley. The New Infinite is supposed to deliver us from that *bête noir* of philosophical speculation, the "endless regress," and, by enabling us to take an infinite multiplicity all at once instead of term by term, to make possible the conception of a *totum simul.*[11] In short, the New Infinite would seem to possess an almost magical virtue.

Now I believe that this apparent victory over the self-contradiction residing in the notion of a realized infinite is a delusion. The self-contradiction has only been concealed—hidden away within the definition of infinity. The surprising dialectical potency of the "New Infinite" results from an ambiguous employment of the notion of totality. For when Dedekind speaks of the endless series of cardinal numbers as a *system* he tacitly imports the notion of totality, and consequently (if the term "totality" is understood in its usual sense) of finitude, into his definition of infinity; since we naturally think of a system as a whole—as a somewhat that is completely given. That the contradiction, instead of being overcome, has merely been concealed from view, appears even more clearly when we consider the phraseology of Cantor. As we have seen, the first example of his "transfinite" aggregate is the "totality of finite cardinal numbers." But he himself speaks of "the unlimited series of finite cardinal numbers"; and, if the series is *unlimited,* by what right is it called a totality?

[8] *Critique de la Doctrine de Kant,* p. 35.

[9] *Hibbert Journal,* Vol. II., p. 540.

[10] *Scientific Method in Philosophy,* pp. 130, 155.

[11] *The World and the Individual,* Vol. I., Supplementary Essay.

Such an employment of terms can only be justified by assuming a peculiar definition of the word "totality." Thus when Cantor speaks of a "totality," he may mean no more than that the collection or series denoted by the word is *determinate, i. e.,* is so defined that it is in principle possible to tell whether or not it includes any given term or collection of terms. For example, we can always tell whether or not a given number belongs to the series of even numbers or to the series of old numbers; and, inasmuch as these series are thus logically distinguishable, there is a sense in which each of them, though by the law of its formation an endless series, is nevertheless a definite and thinkable unity.

Now there are no doubt many such logically distinguishable types of endless series; and it is of course perfectly legitimate for the mathematician to study them, and even to call them "transfinite numbers," if he wishes to employ that terminology and is not himself led astray by it. The difficulty is, however, that some of the more enthusiastic champions of the New Infinite, those especially who have attempted to apply the conception to the solution of problems in theology and philosophy, have given it, at least by implication, a meaning which from the point of view of logic it can not have. They have, in short, forgotten the equivocality of the notion of "totality." For the infinite—whether new or old—can not be regarded as a somewhat that is actually existing, but only as a scheme or plan that is in process of realization; because, understood as an actually existing somewhat, it would be a totality, not merely in the sense of a series defined sufficiently for purposes of identification, but in the sense of a whole, no part of which would be lacking.

In accordance with Professor Cobb's rule of procedure we have been asking about the new definition of infinity, not only, "Is it true?" but also, "What does it mean?" Or rather we have been asking what it must be understood to mean if we are to accept it as logically possible. The same method of procedure must be applied to the definition suggested by Professor Cobb: "A group is said to be infinite when, if a is any finite number that has been chosen, the group has a subgroup of a elements." The "infinite group is chosen before the number a, and the subgroup is chosen after the number a." But what is meant by saying that the "group" is "chosen," "fixed," "given," *etc.?* This, it seems to me, is the crux of the whole question. The "group of finite integers," for example, is a definite unity or totality only in our narrower sense, namely, only in the sense that it is so defined as to be distinguishable from other groups. The "group" which is infinite by the proposed definition is then merely a series which is shown by the proposed test to be an *inexhaustible* and, therefore, *endless* series. And this, I take it, is the manner in

which the definition is interpreted by Professor Cobb himself; for he tells us in the more recent of the two articles referred to above that Kant's "indefinite" is the mathematical infinite.

Thus interpreted, the New Infinite is indeed logically unassailable —and also perfectly harmless. It does not help in the solution of any of the problems of philosophy or theology. It is a shorn Samson.

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REVIEWS AND ABSTRACTS OF LITERATURE

The Validity of the Religious Experience. GEORGE A. BARROW. Boston: Sherman, French & Company. 1917. Pp. 247.

Since the pioneer work of Starbuck and James in the psychology of religion, many similar studies have been made of the religious experience and of religious belief. An older and still a common method is the metaphysical approach to the philosophy of religion, in the effort, first of all, to establish the existence of God and the other objects of religious belief. Dr. Barrow has united the more recent scientific attitude with the older metaphysical method in a philosophical study, but in a study of the religious experience itself. By examining this experience, he seeks to show that the experience has within it positive theological implications. Whether or not his arguments for "the validity of the religious experience" seem cogent, the book deserves a careful reading by all those interested in religious problems.

The so-called religious experience is a fact. Some persons, at some times, unquestionably have the experiences that they call religious experiences. Dr. Barrow begins with the religious experience as a fact, and inquires into its source. "The validity of an experience involves . . . two things, an implication as to the cause, and the truth of the implication" (p. 17). "The claim of religion that it is a relation to a superhuman object or world" (p. 184) may be false, since the cause of the experience may be merely physiological. The religious experience may be only emotion, plus a (false) belief as to the source and significance of the emotion. This question of the truth or falsity of the belief regarding the source of the religious experience is a central one. Though Dr. Barrow says, "It is not belief that we are concerned with, but the religious experience" (p. 157), he elsewhere (p. 41) speaks of the "faith" in a superhuman being which the religious experience contains—and faith is a variety of belief. The "claim of religion" (p. 184) is also a form of belief, otherwise it would not be subject to the categories of truth and falsity.